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# Cubic twistorial string field theory

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Abstract: Witten has recently proposed a string theory in twistor space whose D-instanton contributions are conjectured to compute $\mathcal{N} = 4$ super-Yang-Mills scattering amplitudes. An alternative string theory in twistor space was then proposed whose open string tree amplitudes reproduce the D-instanton computations of maximal degree in Witten's model.

In this paper, a cubic open string field theory action is constructed for this alternative string in twistor space, and is shown to be invariant under parity transformations which exchange MHV and googly amplitudes. Since the string field theory action is gauge-invariant and reproduces the correct cubic super-Yang-Mills interactions, it provides strong support for the conjecture that the string theory correctly computes $N$-point super-Yang-Mills tree amplitudes.

Keywords: Topological Field Theories, Duality in Gauge Field Theories, Chern-Simons Theories, String Field Theory.

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## Contents



## 1. Introduction

In a recent paper [1], Witten showed that the simple holomorphic form of maximal helicity-violating (MHV) tree amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory [2, 3, 4] can be generalized to tree amplitudes with less than maximal helicity-violation. He showed that just as MHV tree amplitudes (i.e. amplitudes with two negative-helicity gluons) are described by curves in twistor space of degree one, tree amplitudes with $d + 1$ negative-helicity gluons are described by curves in twistor space of degree $d$.

Witten then proposed a string theory based on twistor worldsheet variables whose D-instanton contributions were used to compute $\mathcal{N} = 4$ super-Yang-Mills scattering amplitudes. In this string theory, Yang-Mills tree amplitudes with $d+1$ negative-helicity gluons were computed by D-instantons of degree less than or equal to $d$. Since the Yang-Mills tree amplitudes are not described by perturbative amplitudes in the string theory, it is not straightforward to check properties such as factorization and unitarity. This makes it complicated to prove the conjecture that the D-instanton contributions correctly reproduce the super-Yang-Mills tree amplitudes.



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Last month, an alternative string theory based on twistor worldsheet variables was constructed [5] in which $\mathcal{N} = 4$ super-Yang-Mills tree amplitudes were conjectured to be computed by tree amplitudes in the string theory. Assuming that only D-instantons of maximal degree contribute in the model of Witten, these two string theories give the same prescription for super-Yang-Mills tree amplitudes. Note that for MHV amplitudes, it is clear that only D-instantons of maximal degree contribute in Witten's model. And it was recently shown for "googly" amplitudes containing two positive-helicity gluons that only maximal degree D-instantons are needed [6, 7].[1]

In this paper, we construct a cubic open string field theory action for the alternative string theory in twistor space [5]. The cubic action is gauge-invariant and reproduces the usual cubic supersymmetric Yang-Mills interaction terms. Furthermore, it is invariant under parity transformations which exchange MHV and googly amplitudes[2]. Gauge-fixing is straightforward using the $b_0 = 0$ Siegel gauge, so the action can be used to define string Feynman diagrams and compute $N$-point tree amplitudes. Using the standard open string field theory result that the cubic vertex correctly covers moduli space [12, 13, 14], these field theory computations agree with the first-quantized worldsheet theory computations. Since the string Feynman diagrams are expected to be factorizable with the appropriate poles, and since the cubic Yang-Mills amplitudes are correctly reproduced, the field theory action constructed here gives strong evidence for the conjecture that the open string amplitudes correctly compute the $N$-point super-Yang-Mills tree amplitudes.

In section 2 of this paper, the alternative string theory in twistor space is reviewed. In section 3, a cubic twistorial string field theory action is constructed. In section 4, the action is proven to be invariant under parity transformations. And in section 5, we discuss conclusions and possible applications.

## 2. Review of the open twistorial string

### 2.1 Worldsheet action

In the open twistorial string theory proposed in [5], the left-moving worldsheet variables consist of the real twistor variables $Z^I_L = (\lambda^a_L, \mu^{\dot a}_L, \psi^A_L)$ for $a, \dot a = 1$ to 2 and $A = 1$ to 4, their conjugate momenta $Y_{LI}$, and a left-moving current algebra $j^k_L$ for $k = 1$ to $\dim(G)$. Before twisting, the $Z^I_L$ variables have conformal weight zero and the $Y_{LI}$ variables have conformal weight one. One also has the right-moving variables $Z^I_R$, $Y_{RI}$ and $j^k_R$, which satisfy the open string boundary conditions

$$Z^I_L = Z^I_R\,, \qquad Y_{LI} = Y_{RI}\,, \qquad j^k_L = j^k_R. \tag{2.1}$$

Although the formalism resembles heterotic string constructions [15] because of the current

[1] Shortly after our paper was posted on the bulletin board, there appeared a paper [8] proving on-shell parity symmetry and showing that six-point amplitudes with three negative-helicity and three positive-helicity gluons also only require maximal degree D-instantons.

[2] It was recently shown by Edward Witten in independent work that twistor calculations are invariant under parity transformations [10]. There has also been some recent work [11] by Aganagic and Vafa who explain parity symmetry in the context of the model of [9].



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algebra, it seems necessary to work with open strings because the OPE of the vertex operators should lead to the factors $1/(z_i - z_{i+1})$ whose dimension is one, while the total dimension of the closed string vertex operators is two.

Since $Z^I$ and $Y_I$ are real twistor variables, the target space on which the super-Yang-Mills theory is defined has signature $(2,2)$ and the worldsheet has Minkowski signature $(1,1)$. Although one would prefer to have a string theory defined in Minkowski signature $(3,1)$, one can use this string theory to compute super-Yang-Mills scattering amplitudes in signature $(2,2)$, and then Wick-rotate the results to Minkowski signature $(3,1)$. For the case of a $\mathrm{U}(M)$ gauge group, the currents $j^k$ may be written as composite fermionic bilinear operators $\alpha^i\beta_j$ for $i,j = 1\ldots M$, analogous to the 1-5 strings of [1]. But at least at the classical level, the formalism can be generalized to an arbitrary gauge group represented by a current algebra.

The worldsheet action for the matter fields is

$$S = \int \mathrm{d}^2 z \left[Y_{LI}\nabla_{\bar{z}} Z_L^I + Y_{RI}\nabla_z Z_R^I\right] + S_C \tag{2.2}$$

where $S_C$ is the action for the left and right-moving current algebras, and the covariant derivatives

$$\nabla_z = \partial_z - A_z\,, \qquad \nabla_{\bar{z}} = \partial_{\bar{z}} - A_{\bar{z}} \tag{2.3}$$

include a $\mathrm{GL}(1,\mathbb{R}) \equiv \mathrm{GL}(1)$ worldsheet gauge field $A_\mu$ which is defined such that $Y_I$ has $-1$ GL(1) charge and $Z^I$ has $+1$ GL(1) charge.

### 2.2 Physical states

Using the Virasoro and GL(1) generators together with their ghosts $(b,c)$ of conformal weight $(2,-1)$ and $(u,v)$ of conformal weight $(1,0)$, one can construct the BRST operator

$$Q = \int \mathrm{d}z\,[cT + vJ + cu\partial v + cb\partial c] \tag{2.4}$$

where the matter contribution to the stress tensor and GL(1) current is

$$T = Y_I\partial Z^I + T_C\,, \qquad J = Y_I Z^I\,, \tag{2.5}$$

and $T_C$ is the stress tensor for the current algebra. Note that $Q^2 = 0$ when $T_C$ has central charge $+28$, which cancels $c = -26$ of the $(b,c)$ system and $c = -2$ of the $(u,v)$ system. (The variables $Y_I$ and $Z^I$ carry $c = 0$ because of cancellation between the bosons and fermions.) Although tree diagrams can be extrapolated to gauge groups with $c \neq 28$ (much like the Virasoro tree amplitude is well-defined for $d \neq 26$), one expects problems with the $c \neq 28$ current algebras at the loop level.

Physical open string states are described by the GL(1)-neutral dimension-one primary fields

$$V_\phi = j^k\phi_k(Z)\,, \qquad V_f = Y_I f^I(Z)\,, \qquad V_g = \partial Z^I g_I(Z)\,, \tag{2.6}$$

where $\phi_k(Z)$ has zero GL(1) charge, $f^I(Z)$ has $+1$ GL(1) charge and satisfies $\partial_I f^I = 0$, and $g_I(Z)$ has $-1$ GL(1) charge and satisfies $Z^I g_I = 0$. Through the Penrose transform,



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$\phi_k(Z)$ describes $\mathcal{N} = 4$ super-Yang-Mills states and $f^I(Z)$ and $g_I(Z)$ describe $\mathcal{N} = 4$ conformal supergravity states whose twistor description will be explained elsewhere. Note that the conformal supergravity states contribute poles to loop amplitudes and multitrace tree amplitudes, but do not contribute to tree amplitudes involving a single trace where all external states are super-Yang-Mills states. This is because all intermediate states in single-trace tree amplitudes must transform in the adjoint representation of the current algebra. So for computing tree amplitudes involving a single trace, the conformal supergravity states can be ignored.

### 2.3 Tree amplitudes

$N$-point tree-level scattering amplitudes are computed in this string theory by the formula

$$A = \sum_{d=0}^{N-2} \left\langle cV_1(z_1)cV_2(z_2)cV_3(z_3) \int \mathrm{d}z_4 V_4(z_4) \cdots \int \mathrm{d}z_N V_N(z_N) \right\rangle_d \tag{2.7}$$

where $\langle\ \ \rangle_d$ denotes the correlation function on a disk with instanton number $d$. Note that just as the Euler number $\int \mathrm{d}^2z\sqrt{g}R(z)$ is a topological quantity constructed from the worldsheet metric, the instanton number $\int \mathrm{d}^2z\, \epsilon^{\mu\nu}F_{\mu\nu}$ is a topological quantity constructed from the worldsheet gauge field. Since $\int \mathrm{d}^2z\sqrt{g}R = 4\pi$ on a sphere, a gauge field $A_\mu$ with instanton number $\int \mathrm{d}^2z\, \epsilon^{\mu\nu}F_{\mu\nu} = 2\pi d$ can be identified with the spin connection $\Gamma_{\mu\nu}{}^\rho$ in conformal gauge by

$$A_z = \frac{d}{2}\Gamma_{zz}{}^z\,, \qquad A_{\bar{z}} = \frac{d}{2}\,\Gamma_{\bar{z}\bar{z}}{}^{\bar{z}}\,. \tag{2.8}$$

So for instanton number $d$, the worldsheet action $S = \int \mathrm{d}^2z\,(Y_{LI}\nabla_{\bar{z}}Z_L^I + Y_{RI}\nabla_z Z_R^I)$ describes worldsheet fields whose conformal weights are twisted by $-\frac{d}{2}$ times their GL(1) charge, i.e. $(Y_I, Z^I)$ carries conformal weight $(1+\frac{d}{2}, -\frac{d}{2})$.

On a disk with instanton-number $d$, functional integration over the zero modes naively gives the measure factor

$$\langle c\partial c\partial^2 cv\Phi(Z)\rangle_d = \int \mathrm{d}^{8+8d}Z\,\Phi(Z) \tag{2.9}$$

where each $Z^I$ has $d+1$ zero modes. But since the vertex operators for super-Yang-Mills states are GL(1)-neutral and independent of the $v$ ghost, this naively gives zero times $\infty$ where the $\infty$ comes from writing $Z^I = \hat{Z}^I r$ and integrating over the scale factor $r$. One way to regularize (2.9) is to insert the BRST-invariant operator

$$R(z) = v(z)\delta(r(z)-1) + c(z)\partial r(z)\delta(r(z)-1) \tag{2.10}$$

into the correlation function. Since $\partial R = Q(\partial r\,\delta(r-1))$ is BRST-trivial, it is irrelevant where $R$ is inserted on the worldsheet when all external states are on-shell. However, since we will later want to describe a string field theory action involving off-shell states, it is more convenient to regularize (2.9) by working in a "small" Hilbert space where off-shell states are required to be GL(1)-neutral and to be independent of the $v$ zero mode. Functional integration in this "small" Hilbert space is defined by

$$\langle c\partial c\partial^2 c\Phi(Z)\rangle_d = \int Z\mathrm{d}^{7+8d}Z\,\Phi(Z) \tag{2.11}$$



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where $\int Z\mathrm{d}^7 Z$ denotes integration over the projective space $RP^{3|4}$. For on-shell external states, one can easily check that this definition is equivalent to inserting the operator (2.10) into (2.9).

As was shown in [5], the $N$-point tree amplitudes computed using the prescription of (2.7) reproduce the maximal degree D-instanton contribution in the model of [1], and have been conjectured to reproduce $N$-point super-Yang-Mills tree amplitudes. Yang-Mills tree amplitudes are computed as stringy disk diagrams, such as the one in figure 1, using the Yang-Mills vertex operators

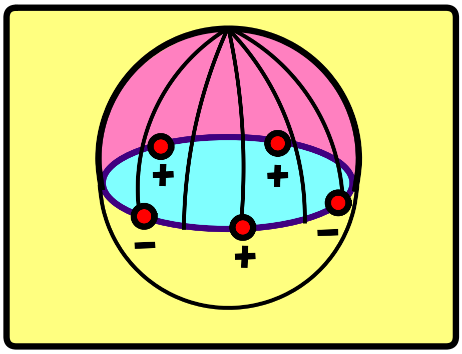

**Figure 1:** The Northern hemisphere represents a disk diagram. In this example, five open string vertex operators are inserted at its boundary. Unlike the models of [1] and [9], the twistor variables in this model are real.

$$\begin{aligned} V_r(z_r) &= j^k(z_r)\delta\left(\frac{\lambda^2(z_r)}{\lambda^1(z_r)} - \frac{\pi_r^2}{\pi_r^1}\right) \times \\ &\quad \times \exp\left(i\frac{\mu^{\dot a}(z_r)}{\lambda^1(z_r)}\bar\pi_{r\dot a}\pi_r^1\right)\phi_{rk}\left(\frac{\psi^A(z_r)\pi_r^1}{\lambda^1(z_r)}\right), \end{aligned} \tag{2.12}$$

where the external momentum is $p_r^{a\dot a} = \pi_r^a\bar\pi_r^{\dot a}$, $\pi_r^a$ and $\bar\pi_r^{\dot a}$ are independent real quantities in signature $(2,2)$,

$$\phi_k\left(\frac{\psi^A\pi^1}{\lambda^1}\right) = (\pi^1)^{-2}\left[A_{+k} + \left(\frac{\psi^A\pi^1}{\lambda^1}\right)^4 A_{-k}\right], \tag{2.13}$$

and the Yang-Mills gauge field is

$$A_{a\dot a k} = \pi_a s_{\dot a} A_{+k} + \bar s_a \bar\pi_{\dot a} A_{-k}$$

where $s_{\dot a}$ and $\bar s_a$ are defined such that $\pi^a \bar s_a = 1$ and $\bar\pi^{\dot a} s_{\dot a} = 1$.

For example, the three-point amplitude gets contributions from the degree zero and degree one correlation functions

$$A = \langle cV_1(z_1)cV_2(z_2)cV_3(z_3)\rangle_{d=0} + \langle cV_1(z_1)cV_2(z_2)cV_3(z_3)\rangle_{d=1}\,. \tag{2.14}$$

The $d=0$ piece contributes to the $(++-)$ amplitude while the $d=1$ term contributes to the $(--+)$ amplitude where the signs indicate helicities. At degree zero, $Z^I(z)$ has zero conformal weight with constant zero modes, i.e.

$$\lambda^a = a^a\,, \qquad \mu^{\dot a} = b^{\dot a}\,, \qquad \psi^A = \gamma^A\,. \tag{2.15}$$

Using GL(1) invariance to gauge $a^1 = 1$, one obtains

$$\begin{aligned} \left\langle \prod_{r=1}^3 cV_r(z_r)\right\rangle_{d=0} &= \int \mathrm{d}a^2 \int \mathrm{d}^2 b^{\dot a}\int \mathrm{d}^4\gamma^A \delta\left(a^2 - \frac{\pi_1^2}{\pi_1^1}\right)\delta\left(a^2 - \frac{\pi_2^2}{\pi_2^1}\right)\delta\left(a^2 - \frac{\pi_3^2}{\pi_3^1}\right)\times \\ &\quad \times \exp\left(ib^{\dot a}\sum_{r=1}^3 \bar\pi_{r\dot a}\pi_r^1\right) f^{k_1k_2k_3}\prod_{r=1}^3 \phi_{rk_r}(\gamma^A\pi_r^1) \end{aligned}$$



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$$
\begin{aligned}
&= \delta\left(\frac{\pi_1^2}{\pi_1^1} - \frac{\pi_2^2}{\pi_2^1}\right)\delta\left(\frac{\pi_1^2}{\pi_1^1} - \frac{\pi_3^2}{\pi_3^1}\right)\delta^2\left(\sum_{r=1}^{3} \bar{\pi}_{r\dot{a}}\pi_r^1\right) \times \\
&\quad \times(\pi_1^1\pi_2^1\pi_3^1)^{-2}\,\mathrm{Tr}\Big((\pi_3^1)^4[A_{+1}, A_{+2}]A_{-3} + (\pi_1^1)^4[A_{+2}, A_{+3}]A_{-1} + \\
&\qquad + (\pi_2^1)^4[A_{+3}, A_{+1}]A_{-2}\Big)\,.
\end{aligned} \tag{2.16}
$$

After scaling $\pi_r^a$ and $\bar{\pi}_r^{\dot{a}}$ in opposite directions until $\pi_1^1 = \pi_2^1 = \pi_3^1$, (2.16) can be written as

$$
\begin{aligned}
\left\langle \prod_{r=1}^{3} cV_r(z_r)\right\rangle_{d=0} &= \delta^4\left(\sum_{r=1}^{3}\bar{\pi}_{r\dot{a}}\pi_{ra}\right)(\bar{\pi}_1^{\dot{a}}\bar{\pi}_{2\dot{a}}) \times \\
&\quad \times \mathrm{Tr}([A_{+1}, A_{+2}]A_{-3} + [A_{+2}, A_{+3}]A_{-1} + [A_{+3}, A_{+1}]A_{-2})\,.
\end{aligned} \tag{2.17}
$$

Note that momentum conservation implies that $(\bar{\pi}_1^{\dot{a}}\bar{\pi}_{2\dot{a}}) = (\bar{\pi}_2^{\dot{a}}\bar{\pi}_{3\dot{a}}) = (\bar{\pi}_3^{\dot{a}}\bar{\pi}_{1\dot{a}})$ when $\pi_1^1 = \pi_2^1 = \pi_3^1$.

At degree one, $Z^I$ has $-1/2$ conformal weight so it has the zero modes

$$
\lambda^a(z) = a^a + \widetilde{a}^a z\,, \qquad \mu^{\dot{a}}(z) = b^{\dot{a}} + \widetilde{b}^{\dot{a}} z\,, \qquad \psi^A(z) = \gamma^A + \widetilde{\gamma}^A z\,. \tag{2.18}
$$

It is convenient to describe these zero modes using the variables $[u_r, b^{a\dot{a}}, \gamma^{aA}]$ where

$$
u_r = \frac{a^2 + \widetilde{a}^2 z_r}{a^1 + \widetilde{a}^1 z_r}\,, \qquad b^{\dot{a}} = a_a b^{a\dot{a}}\,, \qquad \widetilde{b}^{\dot{a}} = \widetilde{a}_a b^{a\dot{a}}, \quad \gamma^A = a_a\gamma^{aA}\,, \qquad \widetilde{\gamma}^A = \widetilde{a}_a\gamma^{aA}\,. \tag{2.19}
$$

After gauging $a^1 = 1$ using GL(1) invariance, the jacobian from going to $(a^2, \widetilde{a}^a, b^{\dot{a}}, \widetilde{b}^{\dot{a}}, \gamma^A, \widetilde{\gamma}^A)$ variables to $(u_r, b^{a\dot{a}}, \gamma^{aA})$ variables is $(u_1 - u_2)(u_2 - u_3)(u_3 - u_1)$. So one finds

$$
\begin{aligned}
\left\langle \prod_{r=1}^{3} cV_r(z_r)\right\rangle_{d=1} &= \int \mathrm{d}^3 u_r \int \mathrm{d}^4 b^{a\dot{a}} \int \mathrm{d}^8\gamma^{aA}(u_1 - u_2)^{-1}(u_2 - u_3)^{-1}(u_3 - u_1)^{-1} \times \\
&\quad \times\delta\left(u_1 - \frac{\pi_1^2}{\pi_1^1}\right)\delta\left(u_2 - \frac{\pi_2^2}{\pi_2^1}\right)\delta\left(u_3 - \frac{\pi_3^2}{\pi_3^1}\right) \times \\
&\quad \times\exp\left(ib^{a\dot{a}}\sum_{r=1}^{3}\bar{\pi}_{r\dot{a}}\pi_{ra}\right) f^{k_1k_2k_3}\prod_{r=1}^{3}\phi_{rk_r}(\pi_{ra}\gamma^{aA}) \\
&= \delta^4\left(\sum_{r=1}^{3}\bar{\pi}_{r\dot{a}}\pi_{ra}\right)(\pi_1^a\pi_{2a})^{-1}(\pi_2^b\pi_{3b})^{-1}(\pi_3^c\pi_{1c})^{-1} \times \\
&\quad \times\mathrm{Tr}\Big((\pi_1^d\pi_{2d})^4[A_{-1}, A_{-2}]A_{+3} + (\pi_2^d\pi_{3d})^4[A_{-2}, A_{-3}]A_{+1} + \\
&\qquad + (\pi_3^d\pi_{1d})^4[A_{-3}, A_{-1}]A_{+2}\Big)\,.
\end{aligned} \tag{2.20}
$$

If one scales $\pi_r^a$ and $\bar{\pi}_r^{\dot{a}}$ in opposite directions until $\bar{\pi}_1^{\dot{1}} = \bar{\pi}_2^{\dot{1}} = \bar{\pi}_3^{\dot{1}}$, conservation of momentum implies that $(\pi_1^a\pi_{2a}) = (\pi_2^\alpha\pi_{3a}) = (\pi_3^\alpha\pi_{1a})$ and (2.20) reduces to the parity conjugate



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of (2.17). One can easily check that the sum of the degree zero and degree one contributions in (2.17) and (2.20) correctly reproduce the three-point Yang-Mills couplings.[3]

For $N$-point tree amplitudes, the formula is

$$
\begin{aligned}
A(\lambda_i, \bar\lambda_i, \psi_i) = & \frac{\int \mathrm{d}^{2d+2}a\, \mathrm{d}^{2d+2}b\, \mathrm{d}^{4d+4}\gamma \int \mathrm{d}z_1 \cdots \int \mathrm{d}z_N}{\mathrm{Vol}(\mathrm{GL}(2))} \times \\
& \times \prod_{r=1}^{N} \frac{1}{(z_r - z_{r+1 \bmod N})} \prod_{r=1}^{N} \delta\left(\frac{\lambda^2(z_r)}{\lambda^1(z_r)} - \frac{\pi_r^2}{\pi_r^1}\right) \exp\left(i \frac{\mu^{\dot a}(z_r)}{\lambda^1(z_r)} \bar\pi_{r\dot a} \pi_r^1\right) \times \\
& \times \mathrm{Tr}\left[\phi_1\left(\frac{\psi^A(z_1)\pi_1^1}{\lambda^1(z_1)}\right)\phi_2\left(\frac{\psi^A(z_2)\pi_2^1}{\lambda^1(z_2)}\right)\cdots\phi_N\left(\frac{\psi^A(z_N)\pi_N^1}{\lambda^1(z_N)}\right)\right]
\end{aligned}
\tag{2.21}
$$

where

$$
\lambda^a(z) = \sum_{k=0}^{d} a_k^a z^k\,, \qquad \mu^{\dot a}(z) = \sum_{k=0}^{d} b_k^{\dot a} z^k\,, \qquad \psi^A(z) = \sum_{k=0}^{d} \gamma_k^A z^k\,,
$$

$(a_k^a, b_k^{\dot a}, \gamma_k^A)$ are the zero modes of $Z^I$ on a disk, and the SL(2) part of GL(2) can be used to fix three of the $z_r$ integrals and reproduce the $(b,c)$ correlation function. When $d=1$ and $d=N-3$, this formula has been verified to give the correct super-Yang-Mills MHV and googly tree amplitudes. Since the above formula will be obtained from the string field theory action of this paper (which is expected to have unitary factorization properties), we consider this strong evidence that the formula gives the correct super-Yang-Mills tree amplitudes for arbitrary helicity-violation.

## 3. Cubic string field theory action

### 3.1 Kinetic term

The first step in constructing a field theory action is to construct a kinetic term whose equation of motion and gauge invariance describe the physical spectrum. The off-shell string field will be described by the wave functional

$$
|\Phi\rangle = \Phi[Y, Z, j, b, c, u, v]|0\rangle \tag{3.1}
$$

where $|0\rangle$ is a ground state satisfying

$$
Z_n^I|0\rangle = Y_{(n-1)I}|0\rangle = j_{n-1}^k|0\rangle = b_{n-2}|0\rangle = c_{n+1}|0\rangle = u_{n-1}|0\rangle = v_n|0\rangle = 0 \qquad \text{for } n>0\,. \tag{3.2}
$$

[3]Since the trace over group theory factors in this string theory comes from current algebra OPE's and not from Chan-Paton factors, changing the order of the vertex operators on the boundary does not alter the order in the trace. This implies that the degree zero correlation function $\langle cV_1(z_1)cV_2(z_2)cV_3(z_3)\rangle_{d=0}$ contributes with opposite sign from the correlation function $\langle cV_2(z_1)cV_1(z_2)cV_3(z_3)\rangle_{d=0}$. So these different cyclic orderings naively cancel each other, which would imply that the $d=0$ term does not contribute to the on-shell three-point amplitude [16]. To get a non-vanishing $d=0$ contribution, one should define an analytic continuation so that switching the order of the vertex operators does not switch the sign. This can be accomplished by multiplying the $d=0$ correlation function by the factor $\mathrm{sign}(\bar\pi_1^{\dot a}\bar\pi_{2\dot a})$. Note that under parity symmetry, this means the $d=1$ correlation function should be multiplied by $\mathrm{sign}(\pi_1^a\pi_{2a})$, which is necessary for converting the standard Jacobian $|(u_1-u_2)(u_2-u_3)(u_3-u_1)|$ to the holomorphic Jacobian $(u_1-u_2)(u_2-u_3)(u_3-u_1)$ which was used in the degree one computation.



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We are using the usual mode expansion for open string worldsheet variables on a strip with $0 \leq \sigma \leq \pi$, e.g.

$$\begin{aligned} Z_L^I(\tau,\sigma) &= \sum_{n=-\infty}^{\infty} Z_n^I e^{in(\tau-\sigma)}\,, \qquad & Z_R^I(\tau,\sigma) &= \sum_{n=-\infty}^{\infty} Z_n^I e^{in(\tau+\sigma)}\,, \\ Y_{LI}(\tau,\sigma) &= \sum_{n=-\infty}^{\infty} Y_{nI} e^{in(\tau-\sigma)}\,, \qquad & Y_{RI}(\tau,\sigma) &= \sum_{n=-\infty}^{\infty} Y_{nI} e^{in(\tau+\sigma)}\,, \end{aligned} \tag{3.3}$$

where $(Z_n^I)^\dagger = Z_{-n}^I$, $(Y_{nI})^\dagger = Y_{-nI}$, and $Y_{nI}Z_m^J - (-1)^{\mathrm{sign}(I)} Z_m^J Y_{nI} = \delta_{m+n}\delta_I^J$.

In addition to the usual requirement that the string field $|\Phi\rangle$ carries $+1$ ghost number, it will also be required that $|\Phi\rangle$ is in the "small" Hilbert space defined in section 2. In other words, $|\Phi\rangle$ must be GL(1)-neutral and independent of the $v$ ghost zero mode, i.e.

$$J_0|\Phi\rangle = 0 \qquad \text{and} \qquad u_0|\Phi\rangle = 0\,. \tag{3.4}$$

However, note that $\Phi$ is allowed to depend on inverse powers of $Z_0^I$ since this dependence is necessary for describing the on-shell twistor wavefunctions for super-Yang-Mills states. So the generic off-shell string field is

$$|\Phi\rangle = \sum_s \phi_s(Z_0) f_s(Z_{-n}^I, Y_{-nI}, c_{2-n}, b_{-1-n}, j_{-n}^k, v_{-n}, u_{-n})|0\rangle \tag{3.5}$$

where $f_s$ is an arbitrary polynomial in oscillators $(Z_{-n}^I, Y_{-nI}, c_{2-n}, b_{-1-n}, j_{-n}^k, v_{-n}, u_{-n})$ for $n > 0$ such that its GL(1) charge cancels the GL(1) charge of $\phi_s(Z_0)$.

Under the above conditions, one can define a kinetic term as

$$S_{\mathrm{kin}} = \langle\Phi|Q|\Phi\rangle \tag{3.6}$$

where $Q$ is the BRST operator defined in (2.4) and $\langle\Phi| = \langle 0|\Phi^\dagger$ is obtained from $|\Phi\rangle$ by hermitean conjugation which switches the signs of all mode indices. Note that the BPZ conjugate bra-vacuum $\langle 0|$ is defined to satisfy

$$\langle 0|Z_n^I = \langle 0|Y_{(n+1)I} = \langle 0|j_{n+1}^k = \langle 0|b_{n+2} = \langle 0|c_{n-1} = \langle 0|u_{n+1} = \langle 0|v_n = 0 \qquad \text{for } n < 0\,. \tag{3.7}$$

As usual, one can also write the kinetic term in (3.6) as $\int \Phi * Q\Phi$ where the functional integral is over all configurations of the first half of the string that is identified with the second half of the string.

The zero mode normalization of the action will be defined in the "small" Hilbert space by

$$\langle 0|c_{-1}c_0c_1\,\phi(Z_0)|0\rangle = \int Z_0 \mathrm{d}^7 Z_0\,\phi(Z_0) \tag{3.8}$$

where $\int Z_0 \mathrm{d}^7 Z_0$ is an integral over $RP^{3|4}$ and $\langle 0|$ is the BPZ conjugate of $|0\rangle$. Note that if one tried to define the zero mode normalization of the action in the "large" Hilbert space involving the $v$ zero mode and the GL(1) scale factor, one would run into the problem that the kinetic term of (3.6) should have ghost-number four. Since $Q$ has ghost number one, this would mean that $|\Phi\rangle$ must carry half-integer ghost number.



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This situation has an analog in bosonic closed string field theory. Naively, the zero mode normalization in bosonic closed string field theory should be

$$\langle 0|c_{-1}c_0c_1\bar{c}_{-1}\bar{c}_0\bar{c}_1|0\rangle = 1 \tag{3.9}$$

where $c$ and $\bar{c}$ are the left and right-moving Virasoro ghosts. However, this would mean that the closed string field $|\Phi\rangle$ in the action $\langle\Phi|Q|\Phi\rangle$ should carry half-integer ghost number. The solution is to work in a "small" Hilbert space where $|\Phi\rangle$ is restricted to be independent of the $(c-\bar{c})$ zero mode and to be neutral under $(L_0-\bar{L}_0)$ rotations, i.e. $(b_0-\bar{b}_0)|\Phi\rangle = (L_0-\bar{L}_0)|\Phi\rangle = 0$. Note that these restrictions on the bosonic closed string field are analogous to the restrictions of (3.4) for the twistorial open string field. In this "small" Hilbert space, one can define the zero mode normalization as

$$\langle 0|c_{-1}c_1(c_0+\bar{c}_0)\bar{c}_{-1}\bar{c}_1|0\rangle = 1 \tag{3.10}$$

so that $\langle\Phi|Q|\Phi\rangle$ is nonvanishing when the closed string field $|\Phi\rangle$ carries $+2$ ghost number.

The kinetic term of (3.6) implies the equations of motion $Q|\Phi\rangle = 0$ and the gauge invariance $\delta|\Phi\rangle = Q|\Omega\rangle$ where $J_0|\Omega\rangle = u_0|\Omega\rangle = 0$. These equations are consistent since $[Q,u_0] = J_0$ and $[Q,J_0] = 0$. One can check that the only states in the cohomology of $Q$ at ghost-number one which satisfy $u_0|\Phi\rangle = J_0|\Phi\rangle = 0$ are

$$c_1 j^k_{-1}\phi_k(Z_0)|0\rangle\,, \qquad c_1 Y_{-1I} f^I(Z_0)|0\rangle\,, \qquad c_1 Z^I_{-1} g_I(Z_0)|0\rangle\,, \tag{3.11}$$

which correspond to the super-Yang-Mills and conformal supergravity vertex operators of (2.6). Although the kinetic term $\langle\Phi|Q|\Phi\rangle$ reproduces the super-Yang-Mills and conformal supergravity spectrum, the equations of motion coming from $Q|\Phi\rangle = 0$ are completely different from the standard super-Yang-Mills and conformal supergravity equations of motion. For example, since the only constraint on the super-Yang-Mills twistor field $\phi_k(Z_0)$ is GL(1)-invariance, it does not even appear in the $\langle\Phi|Q|\Phi\rangle$ kinetic term. However, $\phi_k(Z_0)$ will appear in the cubic interaction term.

### 3.2 Cubic term: the $d = 0$ part

The cubic term in the string field theory action can be determined by the requirements that it preserves a nonlinear version of the gauge invariance $\delta|\Phi\rangle = Q|\Omega\rangle$ and that it reproduces the desired on-shell three-point amplitudes. Since the three-point amplitude involves correlation functions of degree zero and degree one, we will need two cubic terms in the field theory action.

The cubic term of degree zero is easily obtained by using Witten's star product [17] to glue the left and right halves of two string fields to give a third string field $|\Phi\rangle * |\Phi\rangle$. The cubic term of degree zero is

$$S_{d=0} = g\langle\Phi|(|\Phi\rangle * |\Phi\rangle) = g\int \Phi * \Phi * \Phi\,. \tag{3.12}$$

If one does not include the cubic term of degree one, one would have the action

$$S = \langle\Phi|Q|\Phi\rangle + \frac{2}{3}g\langle\Phi|(|\Phi\rangle * |\Phi\rangle)\,, \tag{3.13}$$

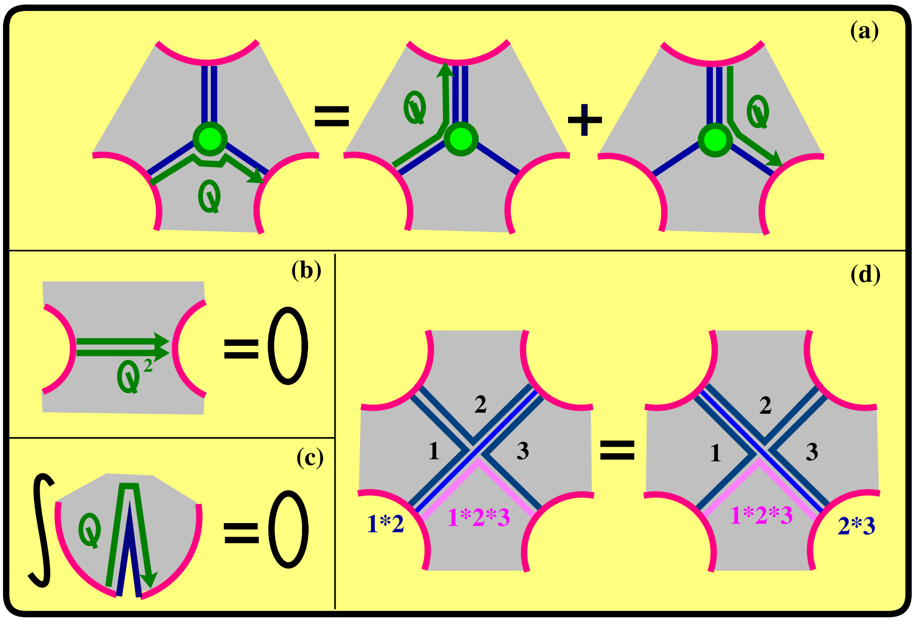


**Figure 2:** Diagrams representing the axioms of string field theory: (*a*) $Q$ is a derivation of the star-product. Note that $u_0$ and $J_0$ are also derivations of the star-product and that a BRST-invariant insertion $F(\pi/2)$ at the midpoint does not spoil the identity. (*b*) Nilpotency of $Q$. (*c*) $\int Q\Phi = 0$. (*d*) Associativity of the star-product.

which has the nonlinear gauge invariance

$$\delta|\Phi\rangle = Q|\Lambda\rangle + g(|\Phi\rangle * |\Lambda\rangle - |\Lambda\rangle * |\Phi\rangle) \tag{3.14}$$

and describes the version of $\mathcal{N} = 4$ super-Yang-Mills with only self-dual interactions [18].

Note that (3.13) is gauge invariant using the usual axioms of open string field theory. Namely, $Q$ is a derivation with respect to the star product, i.e.

$$Q(|\Phi_1\rangle * |\Phi_2\rangle) = (Q\,|\Phi_1\rangle) * |\Phi_2\rangle - |\Phi_1\rangle * (Q\,|\Phi_2\rangle) \tag{3.15}$$

where the minus sign is because $|\Phi\rangle$ is fermionic, $\int Q\Phi = 0$, $Q$ is nilpotent, and the star product is associative, i.e.

$$(\Phi_1 * \Phi_2) * \Phi_3 = \Phi_1 * (\Phi_2 * \Phi_3)\,. \tag{3.16}$$

Since $u_0$ and $J_0$ are integrals of dimension-one currents, they also act as derivatives with respect to the star product. So the constraints $u_0|\Phi\rangle = J_0|\Phi\rangle = 0$ are preserved by the gauge transformation of (3.14) if $u_0|\Lambda\rangle = J_0|\Lambda\rangle = 0$.

### 3.3 Cubic term: the $d = 1$ part

To construct the cubic term of degree one, first note that one can define a BRST-invariant spectral flow operator [19, 20, 21]

$$F(z) = e^{i\sigma(z)} = \exp\left(i\int^z \mathrm{d}y\, Y_I(y) Z^I(y)\right)(1 - ic(z)u(z)) \tag{3.17}$$



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where $\partial\sigma = \{Q, u\} = Y_I Z^I - \partial(cu)$ is the total GL(1) current. Note that $\sigma(y)\sigma(z)$ has no singularity and that $F(z)$ can be expressed in operator language as

$$F(z) = \delta^8(Y(z))(1 - ic(z)u(z))\,. \tag{3.18}$$

Since the GL(1) gauge field $A_{\bar{z}}$ couples to $\partial_z\sigma$ in the worldsheet action, the correlation function on a disk of instanton number $d$ is equivalent to the correlation function on a disk of instanton number zero with $d$ spectral flow operator insertions. To see this, suppose that $\partial_z A_{\bar{z}} = \sum_{r=1}^{d} \delta^2(z - z_r)$ so that the worldsheet field strength is concentrated at points on the worldsheet. Exponentiating the term $-i\int \mathrm{d}^2 z(A_{\bar{z}} J_z)$ in $iS_{\text{worldsheet}}$ therefore gives the contribution $\prod_{r=1}^{d} e^{i\sigma(z_r)}$.

These spectral flow insertions give a background charge to the worldsheet variables which can be mimicked by twisting their conformal weight according to their GL(1) charge. One can think of $\delta^8(Y(z))$ as forcing $Y(z) = 0$ at the insertion point $z$; $Y(y)$ is then proportional to $(y - z)$ near this point, and therefore the dual variable $Z(y)$ is allowed to blow up as $(y - z)^{-1}$ near $z$. The field $Z$ therefore has one new zero mode on the disk, and it can describe a curve of a higher degree.

When all external states are on-shell, the locations of these insertions are irrelevant since $\partial F = Q(iue^{i\sigma})$ is BRST-exact. But in open string field theory, the external states are off-shell, so the locations of these spectral flow insertions are relevant. However, the unique location for these insertions which preserves gauge invariance is the midpoint of the string. So the cubic term of degree one in the action will be defined as

$$S_{d=1} = g'\langle\Phi|F\left(\frac{\pi}{2}\right)(|\Phi\rangle * |\Phi\rangle) = g'\int F\left(\frac{\pi}{2}\right)\Phi * \Phi * \Phi\,. \tag{3.19}$$

The extra insertion of $F(\pi/2)$ does not spoil BRST invariance since $[Q, F(\pi/2)] = 0$.

Since midpoint insertions cause problems [22, 23] in cubic open superstring field theory [24], one might be worried that similar problems could arise here. Fortunately, this does not occur. Unlike the picture-raising operators in the RNS formalism, $F(y)$ has no singularities with $F(z)$ so these insertions do not cause contact term divergences when the midpoints collide [22]. Also, since the kinetic term in the action does not require midpoint insertions, there is no need to truncate out states which are annihilated by the spectral flow operator [23].

So the complete open string field theory action is (see also figure 3)

$$S = \langle\Phi|Q|\Phi\rangle + \frac{2}{3}g\langle\Phi|(|\Phi\rangle * |\Phi\rangle) + \frac{2}{3}g'\langle\Phi|F\left(\frac{\pi}{2}\right)(|\Phi\rangle * |\Phi\rangle)\,, \tag{3.20}$$

which has the nonlinear gauge invariance

$$\delta|\Phi\rangle = Q|\Lambda\rangle + g(|\Phi\rangle * |\Lambda\rangle - |\Lambda\rangle * |\Phi\rangle) + g'F\left(\frac{\pi}{2}\right)(|\Phi\rangle * |\Lambda\rangle - |\Lambda\rangle * |\Phi\rangle)\,. \tag{3.21}$$

This action can be put in a more conventional form by first rescaling the fermionic components of $Z^I$ and $Y_I$ as $\psi^A \to (g'/g)^{\frac{1}{4}}\psi^A$ and $\bar{\psi}_A \to (g/g')^{\frac{1}{4}}\bar{\psi}_A$. Since the norm and $F(\pi/2)$ rescale by a factor of $(g/g')$, the action becomes

$$S = \frac{g}{g'}\langle\Phi|Q|\Phi\rangle + \frac{2g^2}{3g'}\langle\Phi|(|\Phi\rangle * |\Phi\rangle) + \frac{2g^2}{3g'}\langle\Phi|F\left(\frac{\pi}{2}\right)(|\Phi\rangle * |\Phi\rangle)\,. \tag{3.22}$$



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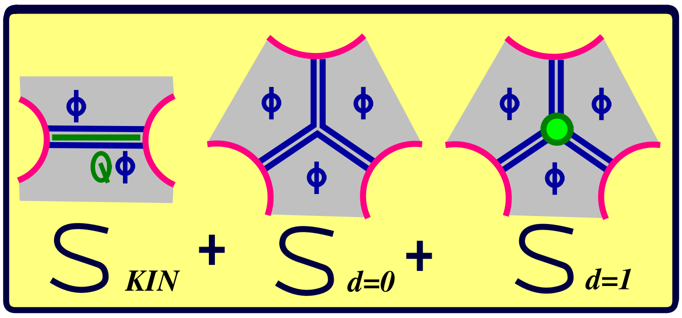


**Figure 3:** A visual representation of the total twistorial string field theory action.

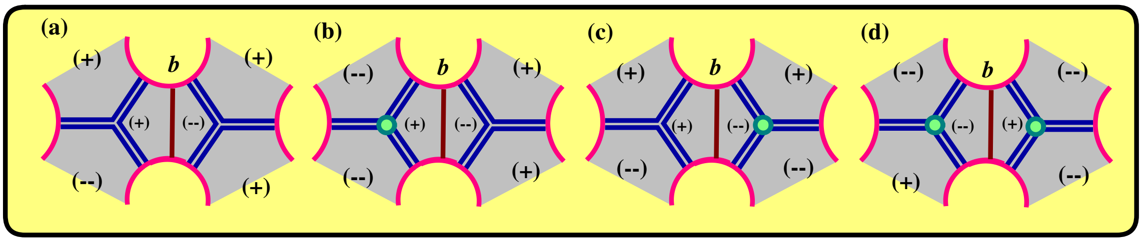


**Figure 4:** A scattering amplitude of four gluons requires two cubic vertices. Each of them is either the $d = 0$ vertex, coupling the $(++-)$ helicities, or the $d = 1$ vertex, coupling the $(--+)$ helicities. $(a)$ Two $d = 0$ vertices, contributing to $(+++-)$. $(b)$ One $d = 0$ and one $d = 1$ vertex, contributing to $(++--)$. $(c)$ One $d = 1$ and one $d = 0$ vertex, contributing to $(--++)$. $(d)$ Two $d = 1$ vertices, contributing to $(---+)$. The exact ordering of the external helicities may differ. The brown line crossing the intermediate string represents an integral of the Virasoro antighost $b(\sigma)$.

Now by rescaling $|\Phi\rangle \to g^{-1}|\Phi\rangle$, one obtains the action

$$S = \frac{1}{gg'}\left[\langle\Phi|Q|\Phi\rangle + \frac{2}{3}\langle\Phi|(|\Phi\rangle * |\Phi\rangle) + \frac{2}{3}\langle\Phi|F\left(\frac{\pi}{2}\right)(|\Phi\rangle * |\Phi\rangle)\right]. \tag{3.23}$$

Note that it is only the product $gg'$ of the two coupling constants that has an invariant meaning. Although the $d = 1$ cubic term looks more unnatural than the $d = 0$ cubic term, we will show in section 4 that the operation of "parity" exchanges the two cubic terms. But let us first demonstrate the equivalence of our string field theory prescription and the first-quantized procedure described in [5].

### 3.4 Equivalence with first-quantized prescription

To define string Feynman diagrams using the action of (3.23), one needs to gauge fix the string field. Since $\{Q, b_0\} = L_0$, this can be done using the standard Siegel gauge-fixing condition that $b_0|\Phi\rangle = 0$. In this gauge, one can use standard open string field theory methods [12, 13, 14] to show that the string Feynman diagrams cover the moduli space of open string tree amplitudes.

A new feature here as compared with bosonic open string field theory is that there are two types of cubic vertices, one which carries instanton number zero and the other which carries instanton number one. For computing $N$-point tree amplitudes of instanton number

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$d$, one gets contributions when $(N-2-d)$ cubic vertices carry instanton number zero and $d$ cubic vertices carry instanton number one. When all external states are on-shell, the choice of which cubic vertices are instanton number zero and which are instanton number one is irrelevant. Naively, this would imply that the on-shell $N$-point tree amplitudes of degree $d$ in equation (2.7) are multiplied by the combinatoric factor $\frac{(N-2)!}{(N-2-d)!\ d!}$. However, as will now be explained, this combinatoric factor is cancelled when one identifies states which are related by the spectral flow operator.

In our string field theory action, the off-shell string field is $\Phi|0\rangle$ where $|0\rangle$ is the ground state defined in (3.2). This ground state is related by the spectral flow operator $F^p$ to other ground states $|p\rangle = F^p|0\rangle$ where $F^p = e^{ip\sigma}$, $\partial\sigma = Y_I Z^I - \partial(cu)$, and $|p\rangle$ satisfies the conditions

$$\begin{aligned} Z^I_{n+p}|p\rangle &= Y_{(n-1-p)I}|p\rangle = 0 \qquad \text{for } n > 0\,, \\ j^k_{n-1}|p\rangle &= b_{n-2}|p\rangle = c_{n+1}|p\rangle = u_{n-1}|p\rangle = v_n|p\rangle = 0 \qquad \text{for } n > 0\,. \end{aligned} \tag{3.24}$$

Note that $Q\Phi|0\rangle = 0$ implies that $Q\Phi|p\rangle = Q\Phi F^p|0\rangle = 0$, so the BRST cohomology constructed from the ground state $|p\rangle$ is isomorphic to the BRST cohomology constructed from the ground state $|0\rangle$. In analogy with the RNS superstring, we will denote states constructed from the ground state $|p\rangle$ as states with "picture" $p$.

When computing tree amplitudes using the string field theory action of (3.23), one only includes intermediate states in the zero picture. This is necessary for unitarity since each physical state should be represented by a unique string field in the BRST cohomology. But when computing tree amplitudes using the first-quantized prescription, functional integration over the worldsheet variables allows all possible intermediate states in all possible pictures. For tree amplitudes in the RNS superstring, this difference between string field theory and first-quantized computations has no effect on scattering amplitudes since picture in the RNS formalism is a conserved quantity. So in RNS tree amplitudes, the picture of the intermediate states is completely determined by the picture of the external states.

But in this twistorial string field theory, picture is not conserved. Since the cubic vertex of degree one involves an explicit $F$ insertion, cubic interactions can violate picture by either *one* or zero. This causes a difference between string field theory and first-quantized computations which cancels the combinatoric factor $\frac{(N-2)!}{(N-2-d)!\ d!}$.

For example, consider the four-point amplitude described by figure 4, and put all external states in the zero picture. In diagram $(a)$, the intermediate state must be constructed from $|0\rangle\langle 0|$ since otherwise one of the two cubic vertices would have picture-violation different from zero or one. But in diagram $(b)$, the intermediate state could be constructed either from $|0\rangle\langle 0|$ or from $|-1\rangle\langle +1|$. In the first case, the cubic vertex on the left has picture-violation one and the vertex on the right has picture-violation zero. And in the second case, the vertex on the left has picture-violation zero and the vertex on the right has picture-violation one. Similarly, in diagram $(c)$, the intermediate state could be constructed either from $|0\rangle\langle 0|$ or from $|+1\rangle\langle -1|$. And in diagram $(d)$, the intermediate state must be constructed from $|0\rangle\langle 0|$.

For on-shell external states, the two types of intermediate states in diagrams $(b)$ and $(c)$ contribute equally. So the string field theory computation (which only includes the



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$|0\rangle\langle 0|$ contribution) is half of the first-quantized computation. As desired, this factor of half cancels the factor of two coming from the combinatoric factor $\frac{(N-2)!}{(N-2-d)!\,d!}$. One can easily check that a similar cancellation occurs for all higher-point tree amplitudes. This is because the number of choices for intermediate states is always equal to the number of ways that the vertices can violate picture, which is equal to the number of ways that the $d=0$ and $d=1$ vertices can be distributed. So the combinatoric factors cancel for any number of external states.

So using the string Feynman diagrams in Siegel gauge, one reproduces the first-quantized prescription of (2.7) for tree amplitudes. Furthermore, it should be possible to show that the poles in the twistorial string Feynman diagrams correspond to physical states and are consistent with unitarity. Since the string three-point amplitudes reproduce the correct cubic interactions, this is strong evidence that the string theory correctly computes the $N$-point super-Yang-Mills tree amplitudes.

## 4. Parity symmetry

One of the characteristic properties of the twistor formalism is that the (left-right) parity symmetry is not manifest. Under this symmetry, the positive and negative helicities are interchanged. While the amplitudes with mostly (+) helicities are described by curves of a small degree, the "googly" amplitudes with mostly (−) helicities require us to consider curves of a large degree which are much more difficult to deal with. Nevertheless, parity is an exact symmetry of the super-Yang-Mills theory S-matrix, and it should be possible to prove this symmetry explicitly.

When the amplitudes are converted to the twistor space, one of the spinors $\lambda^a$ and $\widetilde{\lambda}^{\dot a}$ (usually $\widetilde{\lambda}^{\dot a}$) must be Fourier-transformed. Had we transformed the other spinor $\lambda^a$, we would have obtained the googly description in terms of the dual twistor space. We can check that the external wavefunctions in these two dual pictures are represented by the Fourier transform over all bosonic as well as fermionic twistor coordinates. For example, for the Yang-Mills vertex operator of (2.12), the Fourier transform is

$$\begin{aligned}\widetilde{V}(Y) &= \int d^8Z\, e^{iY_I Z^I} V(Z) = \int d^2\lambda\, d^2\mu\, d^4\psi\, e^{i(\bar\mu_a\lambda^a + \bar\lambda_{\dot a}\mu^{\dot a} + \bar\psi_A\psi^A)} \times \\ &\quad \times j^k \delta\left(\frac{\lambda^2}{\lambda^1} - \frac{\pi^2}{\pi^1}\right) \exp\left(i\frac{\mu^{\dot a}}{\lambda^1}\bar\pi_{\dot a}\pi^1\right)(\pi^1)^{-2}\left[A_{+k} + \left(\frac{\psi^A\pi^1}{\lambda^1}\right)^4 A_{-k}\right] \\ &= j^k\delta\left(\frac{\bar\lambda_{\dot 2}}{\bar\lambda_{\dot 1}} - \frac{\bar\pi_{\dot 2}}{\bar\pi_{\dot 1}}\right)\exp\left(i\frac{\bar\mu_a}{\bar\lambda_{\dot 1}}\pi^a\bar\pi_{\dot 1}\right)\ (\bar\pi_{\dot 1})^{-2}\left[A_{-k} + \left(\frac{\bar\psi_A\bar\pi_{\dot 1}}{\bar\lambda_{\dot 1}}\right)^4 A_{+k}\right],\end{aligned} \tag{4.1}$$

where $Y_I = (\bar\mu_a, \bar\lambda_{\dot a}, \bar\psi_A)$.

Comparing $V(Z)$ of (2.12) with $\widetilde{V}(Y)$ of (4.1), one sees that performing a parity transformation on the states is equivalent to performing a Fourier transform of the vertex operator which switches $Z^I$ with $Y_I$. This is consistent with superconformal transformations since the parity operation exchanges fundamental and antifundamental representations of PSU(2, 2|4). Although the Fourier transformation acts on the function of the zero modes



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of $Z^I$, the stringy completion of this operation will involve the complete interchange of $Z^I$ and its canonical momentum $Y_I$

$$Z^I(\sigma) \;\leftrightarrow\; Y_I(\sigma)\,, \tag{4.2}$$

including the oscillators. In subsection 4.2, we will be more precise how the $Z^I$ and $Y_I$ variables are interchanged.

### 4.1 Parity symmetry of on-shell amplitudes

Before discussing the parity symmetry of the string field theory action, it will be useful to demonstrate that the on-shell amplitude prescription of (2.7) is invariant under parity transformations[4]. Suppose one has an $N$-point amplitude involving $d+1$ negative-helicity gluons and $(N-d-1)$ positive-helicity gluons. Using the prescription of (2.7), this is computed by the correlation function of $N$ vertex operators $V(Z(z_r))$ of (2.12) on a disk of instanton number $d$ where $(Y_I, Z^I)$ has conformal weight $(1+\frac{d}{2}, -\frac{d}{2})$.

Since the spectral flow operator $F = \delta^8(Y)(1-icu)$ can be used as a substitute for instanton number, this amplitude can be equivalently computed with $N$ vertex operators $F(z_r)V(Z(z_r))$ on a disk of instanton number $d-N$ where $(Y_I, Z^I)$ has conformal weight $(1+\frac{d-N}{2}, \frac{N-d}{2})$. But

$$F(z_r)c(z_r)V(Z(z_r)) = \delta^8(Y(z_r))c(z_r)V(Z(z_r)) = c(z_r)\int \mathrm{d}^8 Z\, e^{iY_I Z^I} V(Z(z_r)) \tag{4.3}$$

is the Fourier-transform of $c(z_r)V(Z(z_r))$ defined in (4.1). And since $Y_I$ now has conformal weight $-(N-d-2)/2$, the integration over zero modes involves curves of degree $(N-d-2)$ in $Y_I$.

So the $N$-point amplitude with $d+1$ negative-helicity gluons and $(N-d-1)$ positive-helicity gluons can be computed either using the correlation function of vertex operators $V(Z)$ and degree $d$ curves in $Z^I$, or equivalently, using the correlation function of vertex operators $\widetilde{V}(Y)$ and degree $(N-d-2)$ curves in $Y_I$. These two computations are related by a parity transformation which switches positive and negative helicities and also switches $Y_I$ with $Z^I$.

It will now be shown that this parity invariance of on-shell amplitudes can be understood as coming from parity invariance of the string field theory action.

### 4.2 Off-shell parity: the kinetic term

Let us denote $P$ as the parity operator that is responsible for the interchange of $Z^I$ and $Y_I$. To prove that the kinetic term (3.6) is invariant under parity transformations, we need to show that

$$\langle\Phi|Q|\Phi\rangle = \langle\Phi|P^\dagger Q P|\Phi\rangle \tag{4.4}$$

where $P|\Phi\rangle$ is the parity transform of $|\Phi\rangle$.

[4] This demonstration was inspired by comments of Edward Witten and Warren Siegel on parity symmetry in twistor calculations.



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Since $P$ will be defined to be a unitary transformation, $P^\dagger = P^{-1}$. So we need to show that the BRST operator commutes with $P$, i.e.

$$P^{-1} Q P = Q \,. \tag{4.5}$$

Since $P$ should exchange $Y_I$ with $Z^I$, we shall define

$$P^{-1} Z^I(z) P = P^{IJ} Y_J(z)\,, \qquad P^{-1} Y_I(z) P = P_{IJ} Z^J(z) \tag{4.6}$$

where $P^{IJ}$ and $P_{IJ}$ are constant matrices satisfying $P^{IJ} P_{JK} = \delta^I_K$. To be a unitary transformation, $P$ must preserve the OPE's of $Y_I$ with $Z^J$ which implies that the matrix $P^{IJ}$ is antisymmetric/symmetric when the $IJ$ indices are bosonic/fermionic.

One can check that

$$P^{-1} J(z) P = -J(z)\,, \qquad P^{-1} T(z) P = T(z) - \partial J(z) \tag{4.7}$$

where $J = Y_I Z^I$ and $T = Y_I \partial Z^I$. So to commute with

$$Q = \int \mathrm{d}z (c(T + T_C) + vJ + cb\partial c + cu\partial v)\,,$$

one should define

$$\begin{array}{|l|l|}
\hline
P^{-1} Z^I(z) P \,=\, P^{IJ} Y_J(z)\,, & P^{-1} Y_I(z) P \,=\, P_{IJ} Z^J(z)\,, \\
P^{-1} j^k(z) P \,=\, j^k(z)\,, & \\
\hline
P^{-1} v(z) P \,=\, -v(z) + \partial c(z)\,, & P^{-1} c(z) P \,=\, c(z)\,, \\
P^{-1} u(z) P \,=\, -u(z)\,, & P^{-1} b(z) P \,=\, b(z) - \partial u(z)\,. \\
\hline
\end{array} \tag{4.8}$$

One can verify that the transformations of (4.8) preserve the OPEs of the operators $(Z^I, Y_I, j^k, b, c, u, v)$, so $P$ is a unitary transformation. By defining the parity transformation as in (4.8), one finds that $P^{-1} Q P = Q$, so the kinetic term of (3.6) is parity-symmetric.

### 4.3 Off-shell parity: the cubic terms

What about the cubic terms? We will see that the sum of the two cubic terms is invariant, but the $d = 0$ and the $d = 1$ terms in (3.23) get interchanged. To see this, first note that the star product of two string fields, $|\Phi_1(w)\rangle * |\Phi_2(w)\rangle$, depends in a simple manner on the conformal weight of $w$. If one twists the conformal weight of $w$ by changing its background charge, one finds that [25]

$$|\Phi_1(w)\rangle * |\Phi_2(w)\rangle \to e^{in\sigma(\pi/2)} |\Phi_1(w)\rangle * |\Phi_2(w)\rangle \tag{4.9}$$

where $w = e^{i\sigma}$ and $n$ is the shift in the background charge. Equation (4.9) is easily derived from the fact that all curvature in the cubic vertex is concentrated at the midpoint, so the exponential of the term $n \int \mathrm{d}^2 z\, \sigma(z) R(z)$ in the worldsheet action only contributes at the midpoint.

Under the parity transformation of (4.8), the $d = 0$ cubic term $\langle\Phi|(|\Phi\rangle * |\Phi\rangle)$ transforms into $\langle\Phi|P^{-1}(P|\Phi\rangle * P|\Phi\rangle)$. So if $P|\Phi\rangle * P|\Phi\rangle$ were equal to $P(|\Phi\rangle * |\Phi\rangle)$, the $d = 0$ cubic



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term would be invariant. However, since $P$ does not commute with the stress tensor $T$, it changes the conformal weights of the variables and modifies their star product. Defining

$$T(z) = \{Q, b(z)\} = Y_I \partial Z^I + T_C + b\partial c + \partial(bc) + u\partial v\,, \tag{4.10}$$

one finds that

$$\begin{aligned} P^{-1}T(z)P &= \{Q, P^{-1}b(z)P\} = \{Q, P^{-1}(b(z) - \partial u(z))P\} \\ &= T(z) - \partial(Y_I Z^I - \partial(cu)) = T(z) - \partial^2\sigma \end{aligned} \tag{4.11}$$

where $\partial\sigma = Y_I Z^I - \partial(cu)$.

So the background charge is shifted, which means that

$$P|\Phi\rangle * P|\Phi\rangle = P(e^{i\sigma(\pi/2)}|\Phi\rangle * |\Phi\rangle) = P\left(F\left(\frac{\pi}{2}\right)|\Phi\rangle * |\Phi\rangle\right). \tag{4.12}$$

Therefore, the parity transform of $\langle\Phi|(|\Phi\rangle * |\Phi\rangle)$ is

$$\langle\Phi|P^{-1}P\left(F\left(\frac{\pi}{2}\right)|\Phi\rangle * |\Phi\rangle\right) = \langle\Phi|F\left(\frac{\pi}{2}\right)(|\Phi\rangle * |\Phi\rangle)\,,$$

which is the $d = 1$ cubic term.

Similarly, the $d = 1$ cubic vertex $\langle\Phi|F(\pi/2)(|\Phi\rangle * |\Phi\rangle)$ transforms into

$$\langle\Phi|P^{-1}F\left(\frac{\pi}{2}\right)(P|\Phi\rangle * P|\Phi\rangle)$$

under a parity transformation. Using (4.12), this is equal to

$$\langle\Phi|P^{-1}F\left(\frac{\pi}{2}\right)P\left(F\left(\frac{\pi}{2}\right)|\Phi\rangle * |\Phi\rangle\right).$$

But one can easily check from the transformation of (4.8) that

$$P^{-1}F\left(\frac{\pi}{2}\right)P = F^{-1}\left(\frac{\pi}{2}\right)$$

where $F^{-1}(z) = e^{-i\sigma(z)}$. Since $F^{-1}(\pi/2)F(\pi/2) = 1$, one finds that $\langle\Phi|F(\pi/2)(|\Phi\rangle * |\Phi\rangle)$ transforms into $\langle\Phi|(|\Phi\rangle * |\Phi\rangle)$ under parity, which is the $d = 0$ cubic vertex.

This is exactly what we want: the two cubic terms in (3.23) are interchanged. It agrees with the fact that the $d = 0$ term couples the $(++-)$ helicities while the $d = 1$ term couples the $(--+)$ helicities, and these two cases are $P$ images of each other.

## 5. Conclusions and outlook

We have described the string field theory version of the twistorial open string. The action has a quadratic term, based on the BRST operator, and two cubic terms. One of these cubic terms contains the spectral flow operator that is able to increase the degree of the curve represented by the worldsheet, and the usual string field theory calculation of scattering amplitudes seems to reproduce $\mathcal{N} = 4$ super-Yang-Mills amplitudes transformed into twistor space.

Moreover, our string field theory sheds some new light on the origin of the parity symmetry that seems non-trivial in the twistor variables. It would be interesting to see whether our explanation of parity symmetry can be related to the upcoming papers of [10] and [11]. We would like to list several other interesting open problems:



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- *Field theory prescriptions:* a twistor-related field theory prescription, based purely on degree-one curves connected by a propagator, has recently been shown to reproduce Yang-Mills tree amplitudes [26]. We view these results as intriguing, but do not yet understand how to relate this prescription to our field theory action. Since Yang-Mills states cannot be taken off-shell in our formalism, it is unclear how to reproduce the propagators of [26]. A similar difficulty arises in trying to relate our action to the spinor helicity methods developed by Chalmers and Siegel in [27].

- *Supersymmetric actions:* our field theory action provides a manifestly $\mathcal{N} = 4$ supersymmetric method for computing $\mathcal{N} = 4$ super-Yang-Mills amplitudes, which has been a longstanding open problem using standard superspace approaches. Admittedly, our solution of this problem is highly non-conventional since there is no natural way to take the theory off-shell and since the super-Yang-Mills fields are necessarily coupled to conformal supergravity fields. Nevertheless, our string field theory action may give some useful clues for constructing conventional superspace actions for $\mathcal{N} = 4$ super-Yang-Mills, perhaps by including couplings to conformal supergravity.

- *Loop diagrams:* although we have only investigated tree amplitudes, one can in principle use our string field theory action to compute loop amplitudes. There are several new features which are expected to arise such as anomalies and closed string poles. Hopefully, these new features will help to explain the mysterious $c = 28$ current algebra and the role of conformal supergravity in the open string sector. The question of infrared divergences must also be addressed, which is nontrivial in conformal field theories since they are coupled at all distance scales.

- *Off-shell and nonperturbative physics:* it remains to be seen whether the prescriptions based on twistor variables can be generalized to physics that is off-shell in the usual Minkowski space and whether the twistor space "knows" about non-perturbative physics, for example the S-duality.


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